Calculus of a Single Variable: Examination X. Du 2016

Show all work for credit!

1. In this problem, you will prove that $\frac{d}{dx}\sin x = \cos x$

   a. Show that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Since you are trying to prove $\frac{d}{dx}\sin x = \cos x$, you may not use L'Hôpital's Rule in evaluating this limit as it would already assume that $\frac{d}{dx}\sin x = \cos x$.

   b. Show that $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$. Since you are trying to prove $\frac{d}{dx}\sin x = \cos x$, you may not use L'Hôpital's Rule in evaluating this limit as it would already assume that $\frac{d}{dx}\sin x = \cos x$.

c. Using the limit definition of a derivative, prove that $\frac{d}{dx}\sin x = \cos x$.

d. Given that $\frac{d}{dx}\sin x = \cos x$, evaluate $\frac{d}{dx}\sin^{-1} x$.

2. A sphere has radius r. A cylinder with volume V is inscribed within the sphere such that the cylinder has greatest volume. Find V in relation to r.

3. Water is poured at a constant rate R into a cone-shaped container, which has a radius r and a height h. The cone points towards the ground. Find the height of the water in the cone as a function of time.

4. Use the (ε, δ)-definition of a limit to prove that $\lim_{x \to 3}(x^2 - 1) = 8$.

5. Differentials have many applications in approximation. Use them to approximate:

   a. $\sqrt{80}$

   b. 3.98^3

6. Use the following methods to verify that the product rule is true.

    a. Limit definition of a derivative

    b. Logarithmic differentiation

7. $f(x) = x^5 e^x$

a. Evaluate $\lim_{x \to -\infty} f(x)$.

b. Evaluate the indefinite integral $\int f(x)dx$.

8. Evaluate $\frac{dy}{dx}$ of $4xy + \ln x^2 y = 4$.

9. $f(x) = x^2$.

a. Evaluate $\int_1^3 f(x)dx$ as a limit of a Riemann sum.

b. Check your answer to part (a) by evaluating $\int_1^3 f(x)dx$ as an antiderivative.

10. The equation $25y^2 = 9x^4 - x^6$ forms a Dumbbell Curve.

a. Evaluate $\frac{dy}{dx}$.

b. Find the tangent lines to the curve where $x = 1$.

c. Identify the points on the curve where the tangent line can be expressed as $x = c$, where c is a real constant.

11. Let $f(x) = -4x^2 + 2x + 6$ and $g(x) = e^x$. You may use a graphing calculator.

a. Find the area between the curves.

b. Find the volume of this area rotated about the x-axis.

c. Find the volume of this area rotated about x = 8.

12. Prove that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

13. Solve the differential equation $\frac{dy}{dx} = a^x$.

14. Solve the differential equation $\frac{dy}{dx} = \tan x$.

15. Show that $\frac{dy}{dx} \sinh x = \cosh x$.

16. Use u-substitution to find $\int \csc x dx$.

17. Evaluate $\frac{d}{dx}(x+3)^{x-5}$.

18. Evaluate $\frac{d}{dx}\sqrt[5]{\frac{(2x-1)(x+7)}{5x+9}}$.

19. $f(x) = x^4 - 4x^3 + 5x^2 + 2x - 7$. Use Newton's Method to approximate the values of x accurate to two decimal places such that $f(x) = 4$. Let your original estimates be $x = 2$ and $x = -2$. You may use a graphing calculator.

20. Evaluate $\int \frac{1}{1+e^{x}} dx$.

21. Evaluate $\int \frac{x^{2}+x-16}{(x+1)(x-3)^{2}} dx$.

22. Evaluate $\int \sec^3 x dx$.

23. $f(x) = \sqrt{16-(x-3)^2}$

a. Graph $f(x)$.

b. The domain of $f(x)$ is $x \in [a,b]$. Find a and b.

c. State the range of $f(x)$.

d. Evaluate $f'(x)$.

e. Using trigonometric substitution, evaluate $\int_a^b f(x)dx$.

f. Using basic geometry, verify that your answer to part (e) is true.

24. Evaluate $\int \frac{\sqrt{x^2-16}}{x}dx.$

25. Evaluate $\lim_{x\to 0}[(1+4x)^{\frac{-2}{x}}]$

26. Solve the homogenous differential equation $\frac{dy}{dx} = \frac{2x+3y}{x}$.

27. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$.

28. Solve the first-order linear differential equation $\dfrac{dy}{dx} = \csc x - y \cot x$.

29. Solve the Bernoulli differential equation $\frac{dy}{dx} = xy^4 + 4xy$.

30. Evaluate $\int \sin^2 x dx$.

31. The polar equation $r = 4 + 4\cos\theta$ forms a cardioid. Find its area.

32. $f(x) = e^x$

a. Find the sixth-degree Taylor polynomial for $f(x)$ about $x = 0$.

b. Hence, find a rational approximation for e.

c. Write down e in summation notation as a series to infinity.

Bonus: Given any $m \in \mathbb{Z}^+$, prove by mathematical induction that $\frac{d^n}{dx^n} x^m = \left({}^m P_n\right) x^{m-n}$ is true for all $\{n \in \mathbb{Z}^+ \mid n \leq m\}$.

Bonus: One of the most fundamental equations in complex analysis is Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Prove that Euler's formula is true using power series.